Fractional Factorial Designs

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Unfortunately, the number of runs also grows. Quickly.

Factors (k)	Runs (2^k)
4	16
5	32
6	64
7	128
8	256
9	512

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- Resort to one-at-a-time designs
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- Resort to one-at-a-time designs
- Select only a subset of factors for a factorial design

In both cases we lose the efficiency and power of the factorial design.

A better method is to use a *fractional factorial design*.

Fractional Factorial Designs

A (full) factorial design with k factors, each with two levels, is called a 2^k design.

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We can instead test k factors using only half of the runs of a 2^k design. This is called a 2^{k-1} fractional design.

For example:

- A 2^4 design tests 4 factors using 16 runs.
- A 2^{4-1} design tests 4 factors using 8 runs.
- A 2^3 design tests 3 factors using 8 runs.

Why do fractional factorial designs work?

Fractional designs are motivated by two guiding principles in statistical modeling:

1. *Effect sparsity* states that only a small proportion of the factors in an experiment will have significant effects.
2. *Effect hierarchy* states that lower-order interactions (including primary effects) are more important that higher-order interactions. (This is also called the *hierarchical ordering principle*.)

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Fractional designs rely on an assumption that

$$|\text{low-order effects}| \gg |\text{high-order effects}|$$

Example: the 2^{4-1} fractional design

We begin with a 2^3 full factorial design (the *base design*).

I	A	B	C	AB	AC	BC	ABC
$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$
$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$
$+$	$+$	$+$	$-$	$+$	$-$	$-$	$-$
$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$+$	$+$	$-$	$+$	$-$	$+$	$-$	$-$
$+$	$-$	$+$	$+$	$-$	$-$	$+$	$-$
$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$

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$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$
$+$	$+$	$+$	$-$	$+$	$-$	$-$	$-$
$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$+$	$+$	$-$	$+$	$-$	$+$	$-$	$-$
$+$	$-$	$+$	$+$	$-$	$-$	$+$	$-$
$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$

This design is orthogonal and the design matrix is full rank. We can't add a column for D without messing up these properties.

Confounding

If we choose to set D equal to an existing column in our design, we have *confounded* it. Since the factors vary together in our design we cannot estimate their effects separately.

For example, let D=ABC. Then

$$\beta_{\mathrm{D|ABC}} = \beta_{\mathrm{D}} + \beta_{\mathrm{ABC}}$$

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For example, let D=ABC. Then

$$\beta_{\mathrm{D|ABC}} = \beta_{\mathrm{D}} + \beta_{\mathrm{ABC}}$$

However, by the hierarchical ordering principle we expect that $\beta_{\mathrm{ABC}} \approx 0 \ll \beta_{\mathrm{D}}$, so

$$\beta_{\mathrm{D|ABC}} = \beta_{\mathrm{D}}$$

The 2^{4-1} fractional design (with D=ABC)

We replace the highest interaction (ABC) with D and fill in the rest of the interactions.

I	A	B	C	AB	AC	BC	D= ABC	AD	BD	CD	ABC	BCD	ABD	ACD	ABCD
$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$
$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$+$	$+$
$+$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$+$
$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$+$	$-$	$+$
$+$	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$+$
$+$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
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I	A	B	C	AB	AC	BC	D= ABC	AD	BD	CD	ABC	BCD	ABD	ACD	ABCD
+	−	−	−	+	+	+	−	+	+	+	−	−	−	−	+
+	+	−	−	−	−	+	+	+	−	−	+	+	−	−	+
+	−	+	−	−	+	−	+	−	+	−	+	−	−	+	+
+	+	+	−	+	−	−	−	−	−	+	−	+	−	+	+
+	−	−	+	+	−	−	+	−	−	+	+	−	+	−	+
+	+	−	+	−	+	−	−	−	+	−	−	+	+	−	+
+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

All of the variables are now confounded:

$$\begin{array}{ll} A + BCD & AB + CD \\ B + ACD & AC + BD \\ C + ABD & AD + BC \\ D + ABC & I + ABCD \end{array}$$

Generator Algebra

Filling in the entire design is impractical, especially for large designs. We can identify the *confounding pattern* (or *alias structure*) using a special type of algebra.

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Generator Algebra Axioms

- $XX = X^2 = I$ for any factor X.
- $IX = X$ for any factor X.
- Multiplication commutes, associates, and distributes.

Generating the 2^{4-1} design

We start with the *generator* of the design — the replacement we made to the base design.

$$\begin{aligned} \mathrm{D} &= \mathrm{ABC} \\ (\mathrm{D})\mathrm{D} &= (\mathrm{ABC})\mathrm{D} \\ \mathrm{D}^2 &= \mathrm{ABCD} \\ \mathrm{I} &= \mathrm{ABCD} \end{aligned}$$

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$$
\begin{aligned}
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(\mathrm{D})\mathrm{D} &= (\mathrm{ABC})\mathrm{D} \\
\mathrm{D}^2 &= \mathrm{ABCD} \\
\mathrm{I} &= \mathrm{ABCD}
\end{aligned}
$$

This last statement (I=ABCD) is called the *defining relation* for the design with generator D=ABC.

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With the defining relation (I=ABCD) we can compute the confounding for any variable.

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For A:

$$\begin{aligned} \mathrm{A(I)} &= \mathrm{A(ABCD)} \\ \mathrm{A} &= \mathrm{A^2BCD} \\ &= \mathrm{IBCD} \\ &= \mathrm{BCD} \end{aligned}$$

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For the interaction CD:

$$\begin{aligned} \mathrm{CD(I)} &= \mathrm{CD(ABCD)} \\ \mathrm{CD} &= \mathrm{ABC}^2\mathrm{D}^2 \\ &= \mathrm{AB} \end{aligned}$$

Practice: A 2^{5-1} design

Let's make a 2^{5-1} fractional factorial design (A, B, C, D, & E).

- What is the best generator for this design?

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- Use this generator to construct the defining relation.

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$$\mathrm{EE} = \mathrm{ABCDE}$$
$$\mathrm{I} = \mathrm{ABCDE}$$

- What is the interaction AB confounded with in our design?

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- Use this generator to construct the defining relation.

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- What is the interaction AB confounded with in our design?

$$\begin{aligned} \mathrm{AB(I)} &= \mathrm{AB(ABCDE)} \\ \mathrm{AB} &= \mathrm{A^2B^2CDE} \\ \mathrm{AB} &= \mathrm{CDE} \end{aligned}$$

Next time: Lower fractional factorial designs

A 2^{k-1} fractional factorial design has half the runs of a factorial design.

We can also construct 2^{k-2} designs (1/4 of the runs), 2^{k-3} designs (1/8 of the runs), etc.

These lower fractional designs trade fewer runs for greater confounding. We will develop a metric to characterize the level of confounding.